Spinorial fields

- Most general representation of Lorentz group on fields

$$\psi_a(x) \longrightarrow D_a{}^b(\Lambda)\,\psi_b(\Lambda^{-1}x)$$

$$\downarrow$$

$$(j_-, j_+) \qquad d_{+,-} = (2j_- + 1)(2j_+ + 1) \times (2j_- + 1)(2j_+ + 1)$$

- $J_\pm^i = \dfrac{J^i \pm i K^i}{2}$

$$\begin{cases} [J_\pm^i, J_\pm^j] = i\,\varepsilon^{ijk} J_\pm^k \\ [J_+^i, J_-^j] = 0 \end{cases}$$

$$\boxed{D = e^{-i(\vec{\theta} + i\vec{\eta})\cdot\vec{J}_+ \; - i(\vec{\theta} - i\vec{\eta})\cdot\vec{J}_-}}$$

- simplest case $(0,0) \Longrightarrow J_{\pm}^{i} = 0 \equiv$ scalar field

- next-to-simplest

$$\boxed{\left(\frac{1}{2},0\right) \Longrightarrow \begin{array}{ll} J_{-}^{i} = \frac{\sigma^{i}}{2} & J^{i} = \frac{\sigma^{i}}{2} \\ J_{+}^{i} = 0 & K^{i} = i\frac{\sigma^{i}}{2} \end{array}}$$

$$\boxed{\left(0,\frac{1}{2}\right) \Longrightarrow \begin{array}{ll} J_{-}^{i} = 0 & J^{i} = \frac{\sigma^{i}}{2} \\ J_{+}^{i} = \frac{\sigma^{i}}{2} & K^{i} = -i\frac{\sigma^{i}}{2} \end{array}}$$

$$\Lambda_{L} \equiv D_{(1/2,0)} = e^{-\frac{1}{2}(\underline{\eta} + i\underline{\theta})\cdot\underline{\sigma}} \qquad \Lambda_{R} \equiv D_{(0,1/2)} = e^{-\frac{1}{2}(-\underline{\eta} + i\underline{\theta})\cdot\underline{\sigma}}$$

- Λ_L and Λ_R $\equiv$ finite dimensional <u>non-trivial</u> representations of the Lorentz group

- <u>Notice</u>:

$$\boxed{\Lambda_L^{+} \neq \Lambda_L^{-1}} \quad \boxed{\Lambda_R^{+} \neq \Lambda_R^{-1}}$$

- however:

$$\boxed{\Lambda_L^{+} = \Lambda_R^{-1}}$$

- $\Lambda_{L,R}$ = complex 2×2 with $\det \Lambda_L = \det \Lambda_R = 1$

$$\boxed{\Lambda_{L,R} \in SL(2, \mathbb{C})}$$

- Corresponding fields

- $\left(\frac{1}{2}, 0\right) \equiv$ "left-handed spinor"

  - $\psi_L^{\alpha}(x)$ $\quad \alpha = 1, 2$ $\qquad \psi_L^{\alpha}(x) \longrightarrow \Lambda_L{}^{\alpha}{}_{\beta}\, \psi_L'(\Lambda^{-1}x)$

- $\left(0, \frac{1}{2}\right) \equiv$ "right-handed spinor"

  - $\psi_R^{\dot\alpha}(x)$ $\quad \dot\alpha = 1, 2$ $\qquad \psi_R(x) \longrightarrow \Lambda_R\, \psi(\Lambda^{-1}x)$

- $\Lambda_L = e^{-\frac{1}{2}(\underline{\eta} + i\underline{\theta})\cdot\underline{\sigma}}$ $\qquad \Lambda_R = e^{-\frac{1}{2}(-\underline{\eta} + i\underline{\theta})\cdot\underline{\sigma}}$

- pure rotations : $\vec{\eta} = 0 \implies \Lambda_L = \Lambda_R = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}$

$$\psi_{L,R}(0) \longrightarrow \underbrace{e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\,\psi_{L,R}(0)}$$

same transformation as spin $1/2$ wave-functions you are familiar from QM

$$U_R\,\psi^{\alpha}_{L,R}(0)|0\rangle$$

$$U_R\,\psi_{L,R}(0)\,U_R^{\dagger}\,U_R|0\rangle$$

$$\left(e^{-\frac{i\sigma\cdot\theta}{2}}\psi\right)|0\rangle$$

$$U(g_1)^\dagger \psi_\alpha U(g_1) = D(g_1)^\alpha{}_\beta \psi^\beta$$

$$U(g_2)^\dagger U(g_1)^\dagger \psi U(g_1) U(g_2) = \left(D(g_1) D(g_2)\right)^\alpha{}_\beta \psi^\beta$$

$$|\alpha\rangle = \psi_\alpha |0\rangle \qquad (D^{-1})^\alpha{}_\beta = U$$

$$U(g_1)|\alpha\rangle = D^{-1}(g_1)^\alpha{}_\beta |\beta\rangle = |\beta\rangle D^\beta{}_\alpha{}^*(g_1)$$

$$U(g) v = U(g) v_i |i\rangle = D_{ij} v_j |i\rangle = v_j |i\rangle D_{ij}$$

$$U(g_1) v_\alpha |\alpha\rangle = v_\alpha D^{-1}(g)^\alpha{}_\beta = \left[D^{-1}(g)\right]^T{}_\beta{}^\alpha v_\alpha$$

$$(\Lambda_L^{-1})^T = e^{\frac{1}{2}(\eta + i\theta)\sigma^T}$$

$$= \varepsilon^{-1} e^{-(\eta+i\theta)\sigma} \varepsilon$$

- **Parity** $\quad t \to t \qquad \vec{x} \longrightarrow -\vec{x}$

$$\begin{cases} J^i \sim x^i \partial^j - x^j \partial^i \longrightarrow J^i & \text{axial} \\ K^i \sim t\,\partial^i - x^i \partial_t \longrightarrow -K^i & \text{polar} \end{cases}$$

$$\Rightarrow \quad \boxed{J_+^i = \frac{J^i + iK^i}{2} \overset{P}{\Longleftrightarrow} \frac{J^i - iK^i}{2} = J_-^i}$$

$$\Downarrow$$

$$(j_-, j_+) \Longleftrightarrow (j_+, j_-)$$

$$\psi_L \Longleftrightarrow \psi_R$$

$$e^{A^{-1}BA} = \sum_n \frac{1}{n!}(A^{-1}BA)^n = \sum_n \frac{1}{n!} A^{-1}B^nA = A^{-1}e^{B}A$$

- Rotations:
  - $U(\theta) = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}$
  - conjugate representation $U(\theta)^*$
  - $U(g_1)U(g_2) = U(g_1g_2)$ $\quad U(g_1)^*U(g_2)^* = U(g_1g_2)^*$
  - $U(\theta)^* = A^{-1}U(\theta)A$

- $U(\theta)^* = \left(e^{-\frac{i}{2}\sigma\cdot\sigma}\right)^*$ $\quad * \sim \sigma \to -\sigma^*$

$$= e^{\frac{i}{2}\sigma^*\cdot\theta} = e^{-\frac{i}{2}(-\sigma^*)\cdot\theta} = \varepsilon^{-1}e^{-\frac{i}{2}\sigma\cdot\theta}\varepsilon = \varepsilon^{-1}U(\theta)\varepsilon$$

A

def $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\,\sigma_2$

$\varepsilon_{\alpha\beta}$ $\quad \varepsilon_{12} = 1$

$\varepsilon_{21} = -1$

- $\varepsilon \cdot \varepsilon = -\mathbb{1}$
- $\varepsilon^T = \varepsilon^{-1} = -\varepsilon$
- $\sigma_i^* = \sigma_i^T = \varepsilon\,\sigma_i\,\varepsilon = -\varepsilon^{-1}\sigma_i\,\varepsilon$

$$\boxed{\varepsilon^{-1}\sigma_i\,\varepsilon = -\sigma_i^* = -\sigma_i^T}$$

- $\Lambda_R^T = e^{\frac{1}{2}(\vec{\eta}-i\vec{\theta})\cdot\vec{\sigma}^T} = e^{-\frac{1}{2}(\vec{\eta}-i\theta)\,\varepsilon^{-1}\vec{\sigma}\,\varepsilon} = \varepsilon^{-1}\,e^{-\frac{1}{2}(\eta-i\theta)\cdot\sigma}\,\varepsilon = \varepsilon^{-1}\Lambda_R^{-1}\,\varepsilon$

$$\boxed{\begin{aligned} \Lambda_R^T &= \varepsilon^{-1}\Lambda_L^{\dagger}\,\varepsilon \\ \Lambda_L^T &= \varepsilon^{-1}\Lambda_R^{\dagger}\,\varepsilon \end{aligned}}$$

$\Lambda_R^{-1} = \Lambda_L^{\dagger}$

$\Lambda_L^{-1} = \Lambda_R^{\dagger}$

$$\psi_L \to \Lambda_L \psi_L$$

$$\boxed{\begin{aligned} \Lambda_R &= \varepsilon^{-1} \Lambda_L^* \varepsilon \\ \Lambda_L &= \varepsilon^{-1} \Lambda_R^* \varepsilon \end{aligned}}$$

- $\varepsilon \psi_L^* \xrightarrow{\text{Lorentz}} \varepsilon \Lambda_L^* \psi_L^* = \varepsilon \Lambda_L^* \varepsilon^{-1} \varepsilon \psi_L^* = \Lambda_R \, \varepsilon \psi_L^*$

$$\Longrightarrow \quad \boxed{\begin{aligned} \varepsilon \psi_L^* &\sim \psi_R \\ \varepsilon \psi_R^* &\sim \psi_L \end{aligned}}$$

$$\left(\frac{n}{2},0\right) = \underbrace{\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right)\cdots\left(\frac{1}{2},0\right)}_{\text{symmetrized}}$$

$$\left(\frac{n}{2},0\right)^{*} = \left(\frac{1}{2},0\right)^{*}\otimes\left(\cdots\right)$$

$$= \left(0,\frac{1}{2}\right)\otimes \cdots$$

$$\left(\frac{n}{2},\frac{m}{2}\right) = \left(\frac{n}{2},0\right)\otimes\left(0,\frac{m}{2}\right)$$

$$\psi^{1}_{[\alpha_1}\psi^{2}_{\alpha_2}\cdots\psi^{n}_{\alpha_n]} \equiv T_{\alpha_1\cdots\alpha_n} \equiv \left(\frac{n}{2},0\right)$$

$$\triangle \quad \left(\frac{1}{2},\frac{1}{2}\right) \sim \left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right)$$

- $$\hat{v}^{\alpha\dot{\beta}} \sim \psi_L^{\alpha}\,\psi_R^{\dot{\beta}} = \left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right)$$

$$\hat{v}^{\alpha\dot{\beta}} \longrightarrow (\Lambda_L)^{\alpha}{}_{\gamma}\,(\Lambda_R)^{\dot{\beta}}{}_{\dot{\delta}}\,\hat{v}^{\gamma\dot{\delta}}$$

$$\hat{v} \longrightarrow \Lambda_L\,\hat{v}\,\Lambda_R^{T}$$

- $$v \equiv \hat{v}\,\varepsilon^{-1} \Longrightarrow \boxed{v \longrightarrow \Lambda_L\, v\, \Lambda_L^{\dagger}}$$

$$v'^{\dagger} = (\Lambda_L\, v\, \Lambda_L^{\dagger})^{\dagger} = \Lambda_L\, v^{\dagger}\,\Lambda_L^{\dagger} = \Lambda_L\, v\, \Lambda_L^{\dagger} = v'$$

- $\bar{v} \equiv \hat{v}^T \varepsilon \implies \boxed{\bar{v} \longrightarrow \Lambda_R \bar{v} \Lambda_R^\dagger}$

  $\equiv \varepsilon^{-1} v^T \varepsilon$

- In general $v \in M(2, \mathbb{C}) \sim \mathbb{C}^4$

- . . . but $HM(2,\mathbb{C}) \sim \mathbb{R}^4 \equiv$ invariant subspace $v^\dagger = v$

- $\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3) \equiv (\mathbb{1}, \sigma^i)$ basis of $HM(2,\mathbb{C})$

$$v = v^\mu \sigma_\mu = v^0 - v^i \sigma^i$$

$$= \begin{pmatrix} v^0 - v^3 & -v^1 + i v_2 \\ -v^1 - i v^2 & v^0 + v^3 \end{pmatrix}$$

$$\det v = v_0^2 - v_3^2 - v_1^2 - v_2^2$$

$$= v^\mu v^\nu \eta_{\mu\nu}$$

- similarly $\quad \bar{\sigma}^\mu \equiv \varepsilon^{-1}(\sigma^\mu)^T\varepsilon = (\mathbb{1}, -\sigma^i)$

$$\bar{v} = v^\mu \bar{\sigma}_\mu = \varepsilon^{-1} v^T \varepsilon$$

- $(v^0, v^1, v^2, v^3) \sim \left(\frac{1}{2}, \frac{1}{2}\right)$

- $v \longrightarrow \Lambda_L v \Lambda_L^\dagger = v' \qquad \Lambda_L \in SL(2,\mathbb{C})$

$$\boxed{\det v = \det v'} \qquad \det \Lambda_L = 1$$

Exercise 1

- $$\underbrace{v \longrightarrow \Lambda_L(\theta,\eta)\, v\, \Lambda_L^{+}(\theta,\eta) \iff v^\mu \longrightarrow \Lambda^\mu{}_\nu(\theta,\eta)\, v^\nu}$$

- $$\vec{\theta}, \vec{\eta} \begin{cases} \longrightarrow \Lambda^\mu{}_\nu = \left(e^{-i\vec{\theta}\cdot J + i\vec{\eta} K}\right)^\mu{}_\nu \equiv \mathcal{L}_+^\uparrow \\ \longrightarrow \Lambda_L = e^{-(i\vec{\theta}+\vec{\eta})\cdot\vec{\sigma}} \equiv SL(2,\mathbb{C}) \end{cases}$$

- $$\Lambda_L \sim -\Lambda_L \text{ on } v \implies \mathcal{L}_+^\uparrow = \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$$

$$\Lambda_L \searrow \Lambda \swarrow -\Lambda_L$$

$$\Lambda_L v \Lambda_L^\dagger = (\Lambda v)^\mu \sigma_\mu$$

$$\Lambda_L \underline{v}^\mu \underline{\sigma_\mu} \Lambda_L^\dagger = (\Lambda v)$$

△ Wave equations

≡ covariant eqs. of motion for spinors

contravariant

- $$\boxed{x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}}$$

$$\begin{cases} \eta^{\mu\nu}\eta_{\nu\rho} = \delta^{\mu}_{\rho} \\ \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\,\eta^{\mu'\nu'} = \eta^{\mu\nu} \end{cases}$$

$$\Downarrow$$

- def $\Lambda_{\mu}{}^{\nu} = \eta_{\mu\mu'}\,\eta^{\nu\nu'}\,\Lambda^{\mu'}{}_{\nu'} \Longrightarrow \boxed{\Lambda^{\mu}{}_{\mu'}\,\Lambda_{\rho}{}^{\mu'} = \delta^{\mu}_{\rho}}$

- $\partial_{\nu} \equiv \dfrac{\partial}{\partial x^{\nu}}$

covariant

$$\partial_{\nu} = \frac{\partial x^{\rho'}}{\partial x^{\nu}}\,\partial'_{\rho} = \Lambda^{\rho}{}_{\nu}\,\partial'_{\rho} \Longrightarrow \boxed{\partial'_{\rho} = \Lambda_{\rho}{}^{\nu}\,\partial_{\nu}}$$

$$\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu \quad = \text{contravariant}$$

- $$\partial \equiv \partial_\mu \sigma^\mu = \partial^\mu \sigma_\mu$$

$$\Lambda_L \partial \Lambda_L^\dagger = \partial' \equiv (\Lambda_\mu{}^\nu \partial_\nu)\sigma^\mu$$

- $$\bar{\partial} \equiv \partial_\mu \bar{\sigma}^\mu \implies \Lambda_R \bar{\partial} \Lambda_R^\dagger = \bar{\partial}'$$

- $$\psi_L'(x') = \Lambda_L \psi_L(x)$$

- $$\bar{\partial}' \psi_L'(x') = \Lambda_R \bar{\partial} \underbrace{\Lambda_R^\dagger \Lambda_L}_{\mathbb{1}} \psi_L(x) = \Lambda_R \bar{\partial} \psi_L(x)$$

$$\psi_L = \left(\tfrac{1}{2}, 0\right) \Longrightarrow \bar{\partial}\psi_L = \left(0, \tfrac{1}{2}\right)$$

$$\subseteq \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2}, 0\right) = \left(\tfrac{1}{2} \otimes \tfrac{1}{2}, \tfrac{1}{2} \otimes 0\right)$$

$$= \left(1 \oplus 0, \tfrac{1}{2}\right) = \left(1, \tfrac{1}{2}\right) \oplus \left(0, \tfrac{1}{2}\right)$$

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu}$$

<u>Exercise</u> :)

$\bar{\partial}\,\psi_L$ transforms like right-handed spinor

$$\bar{\partial} \psi_L \equiv \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix} \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_0 \quad \partial_3)\psi_L^1 + (\partial_1 - i\partial_2)\psi_L^2 \\ \\ (\partial_1 + i\partial_2)\psi_L^1 \quad (\partial_0 - \partial_3)\psi_L^2 \end{pmatrix}$$

I) Weyl spinor

Analogue of $\partial F^{\mu\nu} = 0$

$\left(\frac{1}{2},\frac{1}{2}\right)$ $\quad$ $(1,0)\oplus(0,1)$

$$\boxed{i\bar{\partial}\psi_L = 0}$$

$$i(\bar{\partial})_{\dot{\alpha}\alpha}\psi_{L\,\alpha} = 0$$

- $\bar{\partial}\psi_L \equiv \begin{pmatrix} \partial_0+\partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0+\partial_3 \end{pmatrix}\begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix} = 0 \equiv$ two 1st order P.D.E.

- $\partial\bar{\partial}\psi_L = \partial_\nu\partial_\mu\sigma^\nu\bar{\sigma}^\mu\psi_L = \frac{1}{2}\partial_\nu\partial_\mu\left(\sigma^\nu\bar{\sigma}^\mu + \sigma^\mu\bar{\sigma}^\nu\right)\psi_L$

$$= \partial_\mu\partial^\mu\psi_L \equiv \Box\psi_L$$

$$\begin{pmatrix} p_0 + p_3 & p_1 - i p_2 \\ p_1 + i p_2 & p_0 - p_3 \end{pmatrix} \begin{pmatrix} \hat{f}_1(p) \\ \hat{f}_2(p) \end{pmatrix} = 0$$

$$p_0^2 - p_3^2 - p_1^2 - p_2^2 = 0$$

$$\bar{\partial}\psi_L = 0 \quad \Longrightarrow \quad \Box \psi_L = 0$$

Fourier

$$p^\mu p_\mu \, \hat{\psi}(p) = 0$$

$$\Longrightarrow \quad p^\mu p_\mu = 0$$

<u>massless</u> <u>quanta</u>

$$m = 0$$

$\partial_\mu F^{\mu\nu} = J^\nu$

II) Majorana spinor

$$\boxed{i\bar{\partial}\,\psi_L = m\,\varepsilon\,\psi_L^*}$$

- $\bar{\partial}^T = \bar{\partial}^* = \varepsilon^{-1}\partial\,\varepsilon$

- $\varepsilon^{-1}\partial\,\varepsilon\,\psi_L^* = \bar{\partial}^*\psi_L^* = i\,m^*\varepsilon\,\psi_L$

- $i\partial\bar{\partial}\,\psi_L = m\,\partial\,\varepsilon\,\psi_L^* = -i\,m\,m^*\,\psi_L$

$$i\left(\Box + m^2\right)\psi_L = 0$$

Klein-Gordon eq.

Phase rotation symmetry

Weyl

$$\boxed{i\bar{\partial}\psi_L = 0}$$

$\psi_L \to e^{i\alpha}\psi_L \to$ yes

charge $\to$ yes

mass $\to$ no

Majorana

$$\boxed{i\bar{\partial}\psi_L = m\,\epsilon\,\psi_L^*}$$

charge $\to$ no

mass $\to$ yes

$\psi_L \to e^{i\alpha}\psi_L = \psi'$

$$i\bar{\partial}\psi_L' = m\,\epsilon\, e^{2i\alpha}\psi_L'^{*}$$

$\psi_L = e^{-i\alpha}\psi_L'$

III) Dirac spinor

$\psi_L = (\frac{1}{2}, 0)$ $\psi_R = (0, \frac{1}{2})$

charge ✓

mass ✓

$$\boxed{\begin{aligned} i\bar{\partial}\psi_L &= m\,\psi_R \\ i\partial\,\psi_R &= m\,\psi_L \end{aligned}}$$

- $\psi_L \to e^{i\alpha}\psi_L$ $\quad\psi_R \to e^{i\alpha}\psi_R$ $\quad\equiv$ Symmetry

- $\partial\bar{\partial}\psi_L = -im\,\partial\psi_R = -m^2\psi_L \Longrightarrow (\Box + m^2)\psi_L = 0$

$$\boxed{\begin{aligned} i\bar{\partial}\,\psi_L &= m\,\psi_R \\ i\partial\,\psi_R &= m\,\psi_L \end{aligned}}$$

$$\begin{pmatrix} 0 & i\partial \\ i\bar{\partial} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\Longrightarrow \quad i\gamma^\mu \partial_\mu \psi_D = m\,\psi_D$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \equiv \text{Dirac matrices}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \qquad \text{Dirac algebra}$$

$$i\gamma^\mu \partial_\mu \psi_D = m\psi_D \implies (i\gamma^\mu \partial_\mu - m)\psi_D = 0$$

$$(i\not{\partial} - m)\psi_D$$

$$\left.\begin{aligned} \bar\partial \psi_L &= m_1 \psi_R \\ \partial \psi_R &= m_2 \psi_L \end{aligned}\right\} \xRightarrow[\psi_R \to z_R \psi_R]{\psi_L \to z_L \psi_L} \begin{aligned} \bar\partial \psi_L &= m_1 \frac{z_R}{z_L} \psi_R \\ \partial \psi_R &= m_2 \frac{z_L}{z_R} \psi_L \end{aligned}$$

choose $\quad m_1 \dfrac{z_R}{z_L} = m_2 \dfrac{z_L}{z_R} \equiv m =$ real

$$m_1 \rho e^{i\varphi} = m_2 \frac{e^{-i\varphi}}{\rho} = m$$